Lecture 4c: Forming the Density Profile II - Universality?! Pinch – Review, Homogenization and Turbulence Equi-partition Pinch (TEP)

Wenhao Wang

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To give a general picture of the whole class structure, there's a list of key questions that we want to cover if time permit in this class:

- Density peaking
- Zonal flow (v.s. avalenches) and gyro-Bohm breaking
- Intrinsic rotation
- L-H transition, ITB (ion temperature ba) formation
- Edge relaxation - ELMs (edge localized modes), QH (quiescent H-mode) and SOL (scrape-off layer)
- Density limit

Before today's discussion, we'll give a brief review of the last lecture, providing a closure for discussion on the peaked profiles.

1 Review – pinch mechanism of peaked density profile

Today we are going to finish the discussion about the pinch topic, which we started from last lecture. Recently, we had an extensive discussion of ITG/η_i mode, i.e. ion drift-wave turbulence. We talked about the basic derivation of ITG, the flat density case and the shearing flow coupled with ITG. The negative compression is an essential concept for ITG, and we discussed the phenomenological interpretation of how negative compressibility can lead to instability. A lot of materials are posted on the class website if people have an interest for some serious readings.[Link]

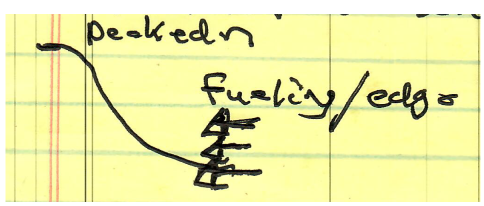


Figure 1: Fueling from plasma edge

We then move to the topic on how to form a peaked density profile with fueling at the edge, since this is one of the key questions in tokamak self-organization. There's an early paper by Strachan, 1982, which discussed the profile formation with intense gas puffing. We may try to understand this starting from the Fick's law,

$$\Gamma = -D\nabla n + Vn \tag{1}$$

To get a steady state with zero total flux, an inward convection $V < 0$ is required to counterbalance the density diffusion ($D > 0$ and $\nabla n < 0$) in tokamak. The ion mixing mode (Coppi and Spight, 1978) is also discussed in previous lecture, and it is used to show how the free energy gradient can drive an inward convective flux, which may explain how a peaked profile can be formed. The process is analogous to Chemotactic response – Keller-Segal model(E.F. Keller and Segal, 1971) – where the chemical concentrations (food supply) may drive up-gradient macroscopic flux of organisms.

The pinch simply refers to an inward convection ($V < 0$), and the ion mixing mode is a prototype for such inward convection. By introducing a phase shift in density perturbation (i.e. non-adiabatic response),

$$\frac{\tilde{n}}{n} = \frac{|e|\phi}{T}\left[1 - i\delta_1 + i\delta_2\right] \tag{2}$$

one can induce a radial flux $\langle \tilde{v}_r \tilde{n} \rangle$. δ_1 is induced by the density gradient ∇n (ω_{*e} term, see previous lecture note) and it's the usual outward, down-gradient flow in ion mixing mode (IMM). $delta_2$ is induced by electron temperature gradient ∇T_e, which is proportional to ∇T_e with the phase of parallel collisional response $\chi_e k_\parallel^2$.

$$\Gamma_{\delta_2} = \langle \tilde{v}_r \tilde{n} \rangle_{\delta_2} \propto \sum_k \frac{|\tilde{v}_{r,k}|^2}{\chi_e k_\parallel^2} \nabla T_e < 0 \tag{3}$$

IMM gives an inward convection, where the fluctuation $\tilde{v}_r$ is produced by ITG mechanism and the gradient ∇T_e provides the phase shift between $\tilde{v}_{r,k}$ and $\tilde{n}$. An analogy between IMM and Keller-Segal model is proposed, that ∇T_e is the analog of nutrient concentration, $|\tilde{v}_{r,k}|^2$ is similar to bacterial energy (random walk step size), and $\chi_e k_\parallel^2$ is the response frequency of sensing.

The entropy production $\frac{dS}{dt} > 0$ is proved to be positive for this process as $\chi_e k_\parallel^2 > \gamma_k, 1/\tau_k$ or $\frac{\omega}{\chi_e k_\parallel^2} \ll 1$. Writing out the entropy production in Onsager matrix,

$$\frac{dS}{dt} = -\left(\frac{\nabla T}{T}, \frac{\nabla n}{n}\right) \begin{pmatrix} \chi_T & -\chi_{cross} \\ -\chi_{cross} & D_n \end{pmatrix} \begin{pmatrix} \frac{\nabla T}{T} \\ \frac{\nabla n}{n} \end{pmatrix} \tag{4}$$

Although the cross term $-\chi_{cross} \frac{\nabla n}{n} \frac{\nabla T}{T}$ is negative, it's on the order of $O(\frac{\omega}{\chi_e k_\parallel^2})$, which is much smaller than the positive diagonal term $\chi_T \left(\frac{\nabla T}{T}\right)^2$. That is, the entropy always increases. The peaked profile gain from pinch may never exceed the loss due to ITG entropy production.

Last but not least, there's limitation for the above discussion, that the electrons must be collisional, saying that the IMM is limited to the edge plasma. Similar approaches have been applied to the core plasma as well, but the discussions depend sensitively on the modes and parameters chosen. Nevertheless, the profile peaking seems to be a universal feature in all these theories, which lead us to today's topic – turbulent equi-partition pinch (TEP).

2 Profile peaking due to geometry (TEP state)

There's a bunch of papers on TEP (Isichenko, Gruzinov and Diamond, 1995; Naulin et al., 1998; Garbet et al., 2005). The "turbulent equi-partition" is based on the theory of homogenization. It is a 2D fluid theory which tries to explain how and why the mixing occurs.

TEP originates from the inhomogeneity of magnetic B field due to the toroidal geometry. In which case, the $E \times B$ drift is compressible, $\nabla \cdot \boldsymbol{v}_{E\times B} \neq 0$. In compressible MHD, the frozen-in condition is given as $\frac{D}{Dt}\left(\frac{\boldsymbol{B}}{\rho}\right) = 0$. Which means, What's really mixed/homogenized in an inhomogeneous field is $\frac{n}{B}$ (i.e. $\frac{f}{B}$ for kinetic theory). After being homogenized (removing gradient),

$$\begin{aligned} &\nabla\left(\frac{n}{B}\right) = \frac{\nabla n}{B} - \frac{\nabla B}{B^2} n = 0 \\ \textbf{i.e.} \quad &\frac{\nabla n}{n} = \frac{\nabla B}{B} \propto -\frac{1}{R} \quad \textbf{(in toroidal geometry)} \end{aligned} \tag{5}$$

That means, there could be a natural state of plasma that the density profile is peaked, proportional to the major radius of the torus. The instabilities cannot flatten the density profile completely, because what should be uniform is $\frac{n}{B}$. Thus, when B is nonuniform, there can be a robust, universal mechanism to allow a peaked profile.

Such profile is mainly due to the geometric property of a toroidal magnetic field, as contrast to ion mixing mode argument where the inward convection is a thermal-electric effect. We will discuss more later why the magnetic field can cause such pinch effect, even though the magnetic field is not a thermal quantity.

However, there're some drawbacks for TEP theory, as the peaking due to toroidal geometry is only modest. If one notice

$$\begin{aligned} -\frac{\nabla T}{T} &\sim \frac{1}{L_T} \gg \frac{1}{a} \\ -\frac{\nabla n}{n} &\sim \frac{1}{R} \end{aligned} \tag{6}$$

the temperature profile is comparatively more peaked than the density profile. Given $\eta_i \geq 1$, it can be difficult for this pinch mechanism to turn the ITG off. It is hard to explain the improved Ohmic confinement (IOC), since in previous lecture we tend to argue that the IOC results from the reduction of η_i due to the peaked density. Another disadvantage is that when coming to Lawson criterion, $n\tau_E T$, increasing density based on TEP may also gives a high density at the edge which is not favorable for plasma confinement. Nevertheless, the negative dependency of gradient $-1/R$ does assure that the plasma concentrates at the center. Though things may get subtle for complicated configurations, like stellerator, one can expect a modest peaking, or a "flattish" n profile in most scenarios based on TEP, which is a typical profile of H-mode.

3 Homogenization

To arrive at TEP, we need to start from the theory of PV/scalar homogenization. It is widely discussed in geophysical fluid dynamics (GFD), but not in plasma physics yet. In this section, we will try to use homogenization to explain the convective mixing of $\frac{n}{B}$ as we showed above. Despite there are various linear instabilities used to explain how the particles are sucked into the center plasma, the universality of "peakedness" can only be explained by turbulent equi-partition. The different modes only tells you how the particles are transported, but TE explains why the profile is peaked.

Prandtl and Batchelor proposed the homogenization theory in the first place, and later it was further developed by Rhines and Young. The homogenization theory also leads to an implication of barriers which will soon be discussed.

Now consider a closed eddy strucutre in a 2D incompressible fluid, i.e. $\nabla \cdot \boldsymbol{v} = 0$. We can write out the vorticity equation, with $\hat{z}$ being the normal direction of the 2D (x, y) plane,

$$\begin{aligned} &\frac{\partial \omega}{\partial t} + \boldsymbol{v} \cdot \nabla \omega - \nabla \cdot \nu \nabla \omega = \boldsymbol{\omega} \cdot \nabla v_z = 0 \\ &\text{where} \qquad \boldsymbol{v} = \nabla \phi \times \hat{\boldsymbol{z}} \end{aligned} \tag{7}$$

The stretching term $\boldsymbol{\omega} \cdot \nabla v_z$ vanishes for 2D fluid, since $\frac{\partial}{\partial z} = 0$. The discussion here does not limit to vorticity ω, but may also apply to other quantities, such as a passively-transport quantity c, the scalar potential in 2D MHD (i.e. flux expulsion problem), and the transport of potential vorticity (PV). The key point is that eq.(7) is just a conservation form with a diffusion term. That is, an operator $\left(\frac{D}{Dt} - \nabla \cdot \nu \nabla\right)$ acting on a scalar quantity. Then, it is obvious that any conserved quantity with certain diffusive mechanism can be written in the form of eq.(7).

For the simplest case, we set $\nu = 0$. As time goes to infinity, the final state satisfies,

$$\nabla \phi \times \hat{\boldsymbol{z}} \cdot \nabla \omega = 0 \tag{8}$$

Apparently, any arbitrary function $\omega = \omega(\phi)$ can be a solution. This suggests that following the closed streamlines, we can develop arbitrarily fine-scale structure across the flux. The field has perfect memory along each streamline, and no inter-streamline mixing occurs in this ideal case ($\nu \to 0$). Such solution is

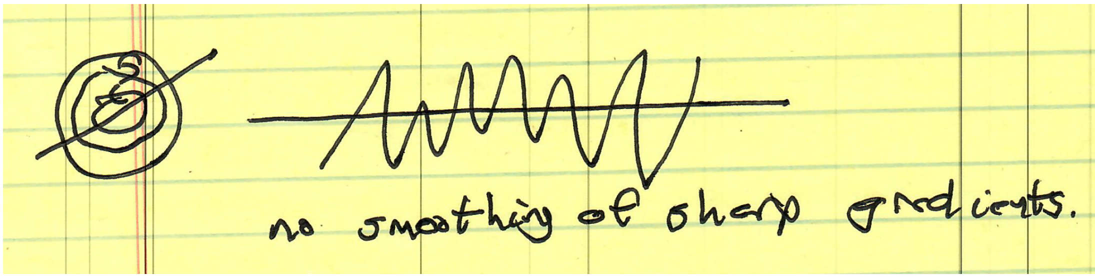


Figure 2: Fine structure across streamlines when $\nu = 0$

obviously unphysical, because any small but finite mixing can cause drastic change of the solution structure. Mathematically, by assuming a finite ν, the first-order differential equation now becomes the second-order differential equation, resulting in a fundamental change in solution properties. As a quote from *Fluid Mechanics* by Landau and Lifshitz,

> Not all solutions of the Navier-Stokes equations are realized in nature.

Like the wake problem in fluid, a small amount of viscosity ν makes a, not only local, but global difference. We will show that instead of having arbitrarily fine structure, we will have a uniform solution $\omega(\phi) \to const.$ as $t \to \infty$.

The proof of this theory was first given by Prandtl and Batchelor. Consider a region of 2D incompressible flow (i.e. vorticity advection) enclosed by a closed streamline C_0 as before. Then, the balance between the convection term and the conservative diffusion term as $t \to \infty$ will assure a uniform vorticity field, i.e. a state of mixing or homogenization. To proof this, we notice $\partial_t \omega \to 0$ as $t \to \infty$, then eq.(7) gives a stationary state

$$\nabla\phi \times \hat{\boldsymbol{z}} \cdot \nabla\omega = \nabla \cdot \nu \nabla \omega \tag{9}$$

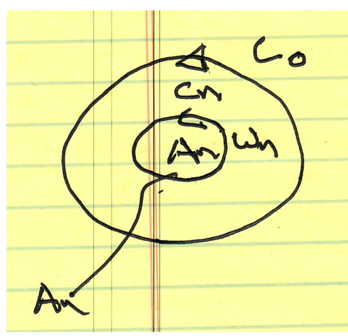


Figure 3: Homogenization within a closed bounding streamline

Now choose another streamline C_n inside C_0. Notice that C_0 is like a separatrix or a boundary for a region that is simply connected (no holes in the region). Thus, any streamline encloses a closed area in the region. The vorticity associated with C_0 is ω_0 and vorticity on C_n is ω_n. The stationary condition assures the vorticity is constant along the streamlines. Integrate over the surface A_n bounded by C_n.

$$\int_{A_n} d^2x \boldsymbol{v} \cdot \nabla\omega = \int_{A_n} d^2x \nabla \cdot \nu \nabla \omega \tag{10}$$

Using the incompressibility and Gauss's law $\int d^2$

$$\int_{A_n} d^2x \boldsymbol{v} \cdot \nabla\omega = \int_{A_n} d^2x \nabla \cdot (\boldsymbol{v}\omega) = \int_{C_n} dl\, \hat{\boldsymbol{n}} \cdot (\boldsymbol{v}\omega) \tag{11}$$

where $\hat{n}$ is the normal direction of the contour. Notice that there's no normal velocity to the streamline, so the integral goes to zero as $\hat{\boldsymbol{n}} \cdot \boldsymbol{v} = 0$. Now we arrive at a meaningful result from eq.(10)

$$0 = \int_{A_n} d^2x \nabla \cdot \nu \nabla \omega = \int_{C_n} dl\ \hat{\boldsymbol{n}} \cdot (\nu \nabla \omega) \tag{12}$$

Again, the Gauss's law is used to simplify the result. Because $\omega = \omega(\phi)$ as we discussed for the case $\nu \to 0$. We can iteratively get the solution for small ν as

$$0 = \int_{C_n} dl\ \hat{\boldsymbol{n}} \cdot \nabla \phi \left(\nu \frac{d\omega(\phi)}{d\phi} \right) \tag{13}$$

After moving the term $\nu \frac{d\omega(\phi)}{d\phi}$ out of the integral (ϕ is constant along streamline), we see the rest is simply the circulation of the velocity $\Gamma = \int \boldsymbol{dl} \cdot \boldsymbol{v} = \int \boldsymbol{dl} \cdot \nabla \phi \times \hat{\boldsymbol{z}} = \int \nabla \phi \cdot \hat{\boldsymbol{z}} \times \boldsymbol{dl} = \int dl\ \hat{\boldsymbol{n}} \cdot \nabla \phi$. Eventually, we have the simple relation

$$0 = \nu \Gamma \frac{d\omega(\phi)}{d\phi} \tag{14}$$

Since the circulation and viscosity are nonzero, we must have $\frac{d\omega}{d\phi} = 0$. C_n is an arbitrary streamline with the boundary C_0, so the result is true for any ϕ value, i.e. $\omega = const.$ is a homogenized solution. In other words, if we have a closed region with finite vorticity, where only incompressible convection and conservative diffusion are allowed, the steady state of this closed region is a homogeneous field.

There are several questions that we didn't discuss in this proof. Firstly, we didn't mention the rate of the system to reach equilibrium. The confinement time of a peaked profile is essential for fusion plasma, so how fast we may reach to such an equilibrium state can be a key question for plasma physicists. Secondly, the Prandtl and Batchelor's proof is novel and instructive. A more standard way for a physicist to think about this question is that one starts from the entropy calculation, and tries to understand the entropy evolution as what is done in quasi-linear theory. Again, we start with the vorticity equation, and notice $\boldsymbol{v} \cdot \nabla \omega = \nabla \cdot (\boldsymbol{v} \omega)$ for incompressible fluid,

$$\begin{aligned} &\frac{\partial \omega}{\partial t} + \nabla \cdot (\boldsymbol{v}\omega) = \nabla \cdot \nu \nabla \omega \\ \Rightarrow \quad &\frac{\partial \langle \omega \rangle}{\partial t} + \frac{\partial}{\partial r} \langle \tilde{v}_r \tilde{\omega} \rangle = \frac{\partial}{\partial r} \nu \frac{\partial}{\partial r} \langle \omega \rangle \end{aligned} \tag{15}$$

This time, by averaging over the θ angle in polar coordinate, we can get a zonal equation as shown in eq.(15). To do so, we separate the vorticity into two parts, $\omega = \langle \omega \rangle + \tilde{\omega}$. $\langle \omega \rangle = \langle \omega \rangle(r)$ is the zonal part that $k_\theta = 0$, and $\tilde{\omega} = \sum_k \bar{\omega}_k(r) \exp(ik_\theta \theta)$ is the perturbation which depends on the wavenumber in θ direction. Obviously, the θ-average gives $\langle \tilde{v}_r \rangle, \langle \tilde{\omega} \rangle = 0$ (note v_r depends on k_θ). Moreover, $\langle \frac{\partial}{r\partial\theta} \nu \frac{\partial}{r\partial\theta} \omega \rangle$ also vanishes as a total integral for the cyclic coordinate θ.

Now, multiply $\langle \omega \rangle$ to the equation and integral over r,

$$\begin{aligned} &\frac{\partial}{\partial t} \int dr \frac{\langle \omega \rangle^2}{2} + \int dr \langle \omega \rangle \frac{\partial}{\partial r} \langle \tilde{v}_r \tilde{\omega} \rangle = \int dr \langle \omega \rangle \frac{\partial}{\partial r} \nu \frac{\partial}{\partial r} \langle \omega \rangle \\ \Rightarrow \quad &\frac{\partial}{\partial t} \int dr \frac{\langle \omega \rangle^2}{2} - \int dr \langle \tilde{v}_r \tilde{\omega} \rangle \frac{\partial}{\partial r} \langle \omega \rangle = - \int dr\ \nu \left(\frac{\partial}{\partial r} \langle \omega \rangle \right)^2 \end{aligned} \tag{16}$$

Here, we do the integral by parts and drop the surface terms, since at the boundary, $v_r|_{r=0,C_0} = 0$. For $\langle \tilde{v}_r \tilde{\omega} \rangle \neq 0$, we can define a turbulence diffusivity, $\langle \tilde{v}_r \tilde{\omega} \rangle = -D_T \frac{\partial}{\partial r} \langle \omega \rangle$, and ν can be extended to an equivalent diffusitity $D = D_T + \nu$. For the trivial solution, we require $D = 0$. But, a more meaningful solution would be $\frac{\partial}{\partial r} \langle \omega \rangle = 0$.

This discussion is quite similar to the quasi-linear theory when we discuss the bump-on-tail problem. The equilibrium distribution satisfies a diffusive equation

$$\begin{aligned} &\frac{\partial}{\partial t} f_0 = \frac{\partial}{\partial v} D \frac{\partial}{\partial v} f_0 \\ \Rightarrow \quad &\frac{\partial}{\partial t} \int dr \frac{f_0^2}{2} = - \int dv\ D \left(\frac{\partial}{\partial v} f_0 \right)^2 \end{aligned} \tag{17}$$

and an equilibrium state can be obtained by either assuming $D = 0$, or $\frac{\partial}{\partial v} f_0 = 0$ in the phase space, i.e. to form a plateau in velocity space.

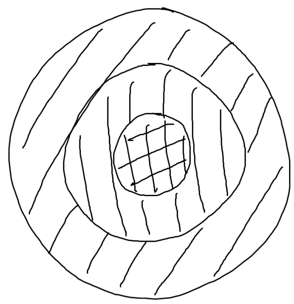


Figure 4: Band-like structure due to spatial structure of viscosity $\nu = \nu(\phi)$

Another remark on this homogenization issue is about the structure of the diffusivity $\nu = \nu(\phi)$. If we have different diffusivity on each streamline, the variation of ν doesn't really affects the results we just discussed. With narrow bands where similar diffusivity presents, they may first form a uniform band region at the same time-scale. Although eventally, the whole region reach to a homogenized state, we can observe these band-like metastable structures if diffisivity ν have a complicated structure over the streamlines. The time required to wash out such band structures are exponentially longer than the homogenization time-scale with the bands. This interesting question is discussed in a paper by Fan X., Diamond P., et al.

4 Homogenization and barriers

In previous section, we discussed the homogenization problem and the key assumptions are (a). a closed, bounding streamline; (b). a simply connected domain; (c). an isolated structure from the surrounding. Homogenization are often associated with a barrier problem. Since a constant field will be formed inside the region, different constants must be separated by borders where a gradient of the constant quantity appears. In the vorticity case, the gradient of vorticity may relates to the gradient of the flow shear. Such strong gradient at the boundary can be a good candidate to study the barrier problem, since we know from our previous lectures that the flow shear can be good to suppress the turbulence and create barriers in plasma.

Recall our discussion on Rossby wave, where the dispersion relation gives $\omega_R = -\frac{\beta k_x}{k^2}$, where $\beta = \frac{d(2\Omega)}{dy}$ is the radial gradient of the background vorticity $2\Omega = 2\Omega_0 + \beta y$. In geophysical fliud, this is called β-plane approximation. y is the meridional distance from the equator to the latitude, which is equivalent to the radial coordinate in our discussion. x is the angular coordinate around the planet (i.e. symmetry direction). Therefore, β can be treated as the gradient between different constant bands. If we have steep gradient at the boundary, we will observe a high Rossby frequency.

In the narrow boundary region where a sharp gradient occurs, the diffusivity D will be modified by the presence of strong Rossby wave ω_{Rk},

$$D \cong \sum_k \langle \tilde{v}^2 \rangle_k \tau_{ck} = \sum_k \langle \tilde{v}^2 \rangle_k \Delta\omega_k^{-1} \to \sum_k \langle \tilde{v}^2 \rangle_k \frac{\Delta\omega_k}{\omega_{Rk}^2 + \Delta\omega_k^2} \tag{18}$$

The mixing through the barrier, characterized by D, decreases as $\omega_{Rk} \gg \Delta\omega_k$ at fixed amplitude. The radial gradient β introduces a resilience against the transport between different constant fields. An interesting question followed is that whether the gradient of vorticity is strong enough to sustain a closed bounding streamline. This leads to the comparison between the vorticity gradient and the fluctuation strength. Dritschel and McIntyre discussed the issue of "Rossby wave elasticity" and "self-sharpening" which is just based on this perspective.

5 Rate of homogenization

Another issure for homogenization process is the time to reach homogeneous state. This requires the interaction between the shear flow and the viscosity. Define the angular coordinate, $y = r\theta$, we can write the shearing velocity,

$$\begin{aligned} \frac{dy}{dt} &= v_y(r) \\ \Rightarrow \quad \frac{d\delta y}{dt} &= \frac{dv_y}{dr}\delta r \end{aligned} \tag{19}$$

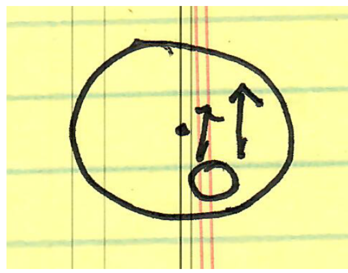


Figure 5: Shear of a rotating flow

The diffusion step size can then be expressed as the integral, $\delta y = \int dt \frac{\partial v_y}{\partial r} deltar$. We can then approximate the diffusion along θ direction as

$$\begin{aligned} \langle \delta y^2 \rangle &= \left(\frac{\partial v_y}{\partial r}\right)^2 \langle \delta r^2 \rangle t^2 \\ &= \left(\frac{\partial v_y}{\partial r}\right)^2 \langle \delta r^2 \rangle t^2 \\ &= \left(\frac{\partial v_y}{\partial r}\right)^2 \frac{D_\nu}{3} t^3 \\ &= \left(\frac{\partial v_y}{\partial r}\right)^2 \frac{\nu}{3} t^3 \end{aligned} \tag{20}$$

Here, we estimate the radial diffusion $\langle \delta r^2 \rangle = \frac{D_\nu}{3} t$, where the factor of 3 characterizes the anisotropic diffusion in radial and circular direction. The mixing length argument then give a mixing time-scale

$$\frac{1}{\tau_{mix}} \cong \left[\frac{1}{\delta y^2}\left(\frac{\partial v_y}{\partial r}\right)^2 \frac{\nu}{3}\right]^{1/3} \tag{21}$$

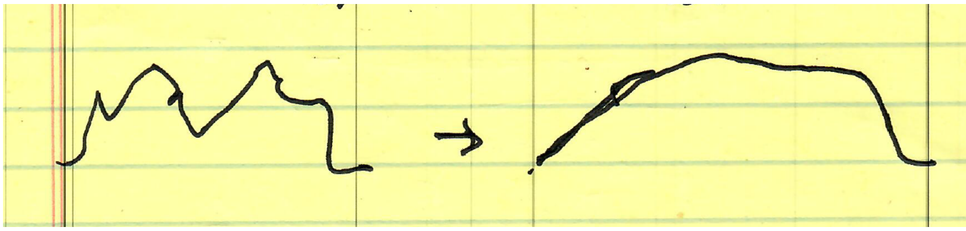
Notice that the angular velocity $\omega_{rot} \sim \frac{\partial v_y}{\partial r} \sim \frac{v_y}{\delta r}$ and $\delta y \sim (\omega_{rot}\delta r)\tau_{mix}$, we can rewrite the mixing time in terms of Reynolds number $Re = \frac{v_y \delta y}{\nu}$,

$$\tau_{mix} \cong \frac{(3Re)^{1/2}}{\omega_{rot}} \sim Re^{1/2}\tau_{rot} \tag{22}$$

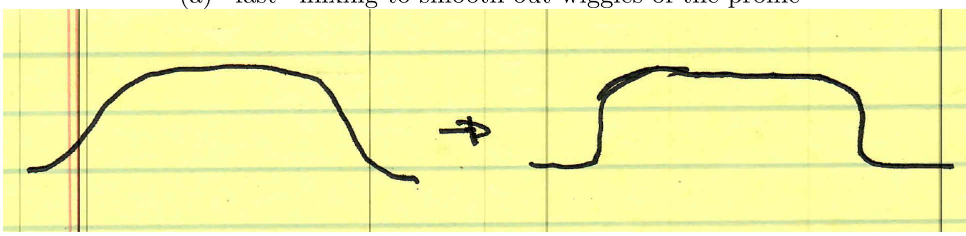
For a purely diffusive process, where $\delta r \sim \delta y$, the diffusion term gives

$$\frac{1}{\tau_{diff}} \sim \frac{\nu}{\delta r^2} \sim \frac{\omega_{rot}}{Re} \tag{23}$$

There are two time-scales in the mixing process: the "fast" mixing comes from the shearing which smooth out the fine structures; the "slow" mixing due to the diffusion sharpens the gradient at the boundary and reach the uniform state.



(a) "fast" mixing to smooth out wiggles of the profile



(b) "slow" mixing to global homogenization and sharpen the gradients

Figure 6: Different time-scales of mixing process

6 convective mixing with non-uniform geometry

Now move back to plasma physics with a non-uniform magnetic field. The continuity equation now gives

$$
\begin{aligned}
&\frac{\partial n}{\partial t} + \nabla \cdot (\boldsymbol{v} n) = 0 \\
\text{where} \qquad &\boldsymbol{v} = \boldsymbol{v}_E = -\frac{c}{B(x)} \nabla \phi \times \hat{\boldsymbol{z}}
\end{aligned} \tag{24}
$$

In tokamak, the magnetic field can be expressed as a function of major radius, $\boldsymbol{B} = B_0(R)\hat{\boldsymbol{z}}$, and the $E \times B$ drift velocity is compressible. Substitute the velocity and note $\nabla \cdot \nabla \phi \times \hat{\boldsymbol{z}} = \nabla \cdot \nabla \times \phi \hat{\boldsymbol{z}} = 0$,

$$
\begin{aligned}
&\frac{\partial n}{\partial t} - \frac{c}{B} \nabla \phi \times \hat{\boldsymbol{z}} \cdot \nabla n + \frac{c}{B^2} \nabla \phi \times \hat{\boldsymbol{z}} \cdot \nabla B = 0 \\
\Rightarrow \qquad &\frac{\partial n}{\partial t} - c \nabla \phi \times \hat{\boldsymbol{z}} \cdot \nabla \left(\frac{n}{B} \right) = 0
\end{aligned} \tag{25}
$$

This looks similar to the incompressible advection we discussed in eq.(7), though the velocity is different from $E \times B$ drift. We can then prove that $\frac{n}{B}$ will be homogenized when we reach the equilibrium. Again, extracting the zonal flow part as we did for eq.(15)

$$
\frac{\partial \langle n \rangle}{\partial t} + \frac{\partial}{\partial r} \langle \tilde{v}_r \frac{\tilde{n}}{B} \rangle = 0 \tag{26}
$$

To focus on the key points of current discussion, we now make a handwavy statement, that there exists a general relation between density perturbation and the velocity, $\frac{\tilde{n}}{B} = -\tilde{v}_r \mathcal{R} \nabla \langle \frac{n}{B} \rangle$, where $\mathcal{R}$ is the response function which depends on the detailed instabilities. In general, this statement means that the perturbation is induced by the gradient of overall geometry. Then, we can calculate the flux term as

$$
\begin{aligned}
\Gamma_n = \langle \tilde{v}_r \frac{\tilde{n}}{B} \rangle &= \sum_k (\tilde{v}_r)_k^* \frac{\tilde{n}_k}{B} \\
&= -\sum_k |(\tilde{v}_r)_k|^2 \mathcal{R}_k \nabla \langle \frac{n}{B} \rangle \equiv -D \nabla \langle \frac{n}{B} \rangle
\end{aligned} \tag{27}
$$

The equivalent diffusivity D is a similar concept when we discuss eq.(16), where the turbulence induces a flux characterized by D_T. Now the discussion from here becomes standard, that we want to analyze the equilibrium condition of a mean-field equation with a diffusive term.

$$\begin{aligned} &\frac{\partial\langle n\rangle}{\partial t} - \frac{\partial}{\partial r} D \frac{\partial}{\partial r}\frac{\langle n\rangle}{B} = 0 \\ \Rightarrow \quad &\frac{\partial}{\partial t}\int dr \langle n\rangle^2 = -\int dr\ D\left(\frac{\partial}{\partial r}\langle n\rangle\right)\left(\frac{\partial}{\partial r}\frac{\langle n\rangle}{B}\right) \end{aligned} \tag{28}$$

To achieve a stationary state, the zero entropy production requires:

(a). $D = 0$: a trivial solution where no turbulence presences.

(b). $\frac{\partial}{\partial r}\langle n\rangle = 0$: a flat density solution, which is not favored for fusion reaction, since we want plasma to concentrate in the center.

(c). $\frac{\partial}{\partial r}\frac{\langle n\rangle}{B} = 0$: this is exactly the fully-mixed (or "canonical") turbulent equi-partition pinch (TEP) we talked about. A radially-decreasing profile is achieved with the dependency $\frac{1}{\langle n\rangle}\frac{\partial\langle n\rangle}{\partial r} = \frac{1}{B}\frac{\partial B}{\partial r} \propto -\frac{1}{R}$.

This peaked profile is solely resulted from the turbulent mixing and the geometry effects, nothing more than the homogenization of $\frac{n}{B}$. This conclusion is independent of the details of instability. That is, the exact form of the equivalent diffusivity D is not so important, as long as the turbulence is driven by the gradient of homogenized quantity, $\frac{\langle n\rangle}{B}$.

Two more comments on TEP peaking mechanism:

(a). Though B is not a thermodynamic variable, its geometric variation influences the threshold of instability. Garbet, et al., 2005 discussed the entropy production with such geometric gradient, and he found that the gradient of thermodynamic parameters must be larger than $1/R$, in order to have a non-zero flux, i.e. to excite the instability. The TEP state puts a load on the energy balance, which is similar to the zonal flow. It would be interesting to see how to form a pinch when a strong zonal flow also presents (Dimits shift regime). The entropy production should be reconsidered in this case.

(b). For kinetic formulation, instead of talking about $\frac{n}{B}$ homogenization, we can focus on $\frac{f}{B}$. In this 2D kinetic version, homogenization issue may be viewed as trapped particle problems. A detailed discussion of this is provided in Isichenko, Gruzinov, Diamond et al., 1996.

As a summary for our discussion on profile peaking, there are two mechanisms that we went through in this class. One is the thermoelectric effects, that the gradient of free energy drives an inward flux to maintain a peaked profile. This approach can generate a very strong density profile, but it can be very sensitive to the modes and models we used. Another mechanism is TEP, which is a universal approach. The peaked profile is driven by the geometric profile, but the peaking can be very weak with a radial dependency of $1/R$.

One of the open questions in this topic is the TEP theory for marginal states. Our current discussion mainly focus on the diffusive flux, i.e. the flux relays on gradient of homogenized quantity, and the diffusivity D can be important. When the instabilities are marginal, a non-diffusive flux occurs which requires a different argument for the peaking process. Another interesting question is the theory of profile peaking in stellerators, where a more twisted geometry of helical fields and toroidal fields gives different scale lengths and trapped particle species, which may introduce extra complication of TEP theory.